<u>Eight Things That Might Just Help You Get a Five on the AP Calculus Exam</u>

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There was a time, perhaps around 1990, when it was possible to predict with some certainty at least the nature and type of free response questions that would be on the AP Calculus exam. The uncertainty generally revolved around fairly mundane issues: Would the particle motion problem start with the position function and ask for velocity and acceleration, or would the acceleration/velocity be given as a starting point? Would the area and volume problem involve two polynomials or would it be a trigonometric function? Would the curve analysis result in two extrema and a point of inflection or only one extremum? Maybe this is an oversimplification, but not by very much. The exam was something that could be predicted and prepared for in quite specific ways.

The AP Calculus exam since 2000 had been, for a number of years, quite a different species. Suddenly there were amusement park rides, rock concerts and snow on a driveway, to name just a few new wrinkles in the very successful effort to make AP Calculus a more appealing course with wider application and utility. Now, though, even the novelty of those problems has settled into some predictable patterns that students who wish to be successful could focus their practice and study on.

Following are eight topics and parts of topics that students should at least feel confident of handling when they appear on the free response exam. There is no guarantee, of course, that problems exactly like these will appear in a given year, and, in addition, these more fully reflect the free response type of items. Multiple choice questions, where students are more apt to be asked to demonstrate command of skills and content knowledge in the calculus course, have remained less changed over the years.

1. <u>Establishing Continuity of a Function at a Value.</u>

Several times in recent years, students have been asked to use limits – or more broadly, the definition of continuity – to establish that a function was or was not continuous at a given value of *x* or *t*. Students appear to have a highly intuitive sense of how to answer that question. Still, many capable students neglect to include all of the essential elements in this process, sometimes perhaps because it appears too transparently true to provide more detail, sometimes perhaps because they have not remembered that there are ordinarily three elements to establishing continuity.

Most often, when the problem is posed in a piecewise function, we expect the student to do these three things:

a) Compute the left and right hand limits
b) Compute the function value
c) Validate that the result from (a) equals the result from (b)

<u>Example 1</u>:

Let *f* be a function defined by:

$$f(x) = \begin{cases} \cos(\frac{\pi}{4}x) + 1, & x \le 2 \\ \frac{x^2}{2} + x - 3, & x > 2 \end{cases}$$

Show that *f* is continuous at $x = 2$.

2. <u>Motion Revisited.</u>

Although motion problems have been a staple on the AP Calculus exam for decades, there are some new trends that have shown up in recent years, mostly having to do with the availability of handheld calculators and a renewed emphasis on justifying answers. Below are some of the key issues that have been tested in recent years at least once and in most cases several times:

a) **Is the speed increasing or decreasing?**
   This question comes from the physics side of the calculus curriculum and, although it is fairly simple to answer yes or no, a justification is required. The justification rests on recognizing something about the sign of the velocity and acceleration, namely, that when velocity and acceleration are like signed, speed is increasing and, when they are unlike signed, speed is decreasing. Students generally find it easy to visualize that a particle moving right with positive acceleration has increasing speed. It is less intuitive but nevertheless accessible for students to imagine that a particle which is moving left with negative acceleration has increasing speed. The example of a falling object does this quite well. Since this question often is only worth one point, the point requires both an answer and a justification. Typically, the question will be linked to evaluating acceleration at a point, a calculation that students can often do in the calculator portion of the exam. The stem for this type of question might look something like the following example:

<u>Example 2:</u>

A particle moves in a straight line with velocity given by the function $v(t)=t^3e^{-t}-1 \; for \; t\geq 0$. Find the acceleration at time t = 4. Is the speed of the particle increasing or decreasing at t = 4?

b) **How far did the particle travel over a given interval of time and with what average velocity?**

This question once required much more algebra and calculation than it does now, provided that it occurs in the calculator portion of the exam. At one time, the student needed to determine the time at which the particle changed direction and then compute the absolute value of the integral over each subinterval. Now, it is possible to get this value by simply calculating $\int_a^b |v(t)| dt$ for a given velocity over a given interval [*a, b*] and then divide by the length of the interval to compute the average velocity. Taking the absolute value of the velocity first essentially renders the motion in one direction. Students who do not see this simple maneuver are apt to do much more work to find the zero(es) of the velocity function. Here is a typical problem:

<u>Example 3:</u>

A particle moves along the x-axis with velocity given by the function $v(t) = (t-2)\sin(e^t)$ for $t \geq 0$

A. How far did the particle travel over the interval $0 \leq t \leq 2$?

B. What was the particle's average velocity on that interval?

c) **What is the position of the particle at a specific time *t*, given an initial position and velocity?**

This is one of the many ways that the Fundamental Theorem of Calculus manifests itself in the AP Calculus course and exam. Variously referred to as "Part I" or "The Evaluation Part" of the FTC, it is usually stated in terms like this:

Suppose that f(x) is continuous on an interval [a, b] and that F(x) is an antiderivative of f(x) on [a, b]. Then $\int_a^b f(x)dx = F(b) - F(a)$

An analogous statement, linking *f(x)* and *f'(x)* rather than *f(x)* and *F(x)*, is the following: $\int_a^b f'(x)dx = f(b) - f(a)$, which can be easily restated to this:

$f(b) = f(a) + \int_a^b f'(x)dx$. It is now a simple matter to transpose this to the motion situation, giving us a very valuable relationship in calculus:

Let v(t) be the velocity of a particle moving in a straight line over an interval of time $t \in [a,b]$ *such that, at time t = a, its position is given by x(a) = k. Then its position at time t = b is given by:* $x(b) = k + \int_a^b v(t)\,dt$. Let's look once again at an example. Note in the example that the velocity does not have a closed form antiderivative, so the student cannot work exclusively with the position function but must appeal to the FTC.

<u>Example 4:</u>

Over the interval $0 \le t \le 3$, a particle moves along the *x*-axis with velocity given by: $v(t) = \sin(t^2)$. Given that its position, *x(t)*, at time *t* = 0 is 4, what is its position at time *t* = 3?

3. Net Change as the Integral of a Rate.

Computing final position from an initial position and a velocity is a specific example of a generalized result of the FTC, namely that any rate of change can be integrated on an interval to find the amount of change on that interval. In motion problems, the paradigm is very well understood by students because they deal with motion all the time. But when the rate of change provided is not so familiar – heating or cooling water, people entering an auditorium, birds passing during migration, and a host of others – the student must change the scene and effectively translate the situation from known to unknown. Further complicating the picture is that this question is often accompanied by questions of interpretation and meaning. Specifically, when given a rate function, $R(t)$, students are likely to be asked to explain the meaning of both $\int_a^b R(t)\,dt$ and $\frac{1}{b-a}\int_a^b R(t)\,dt$. It is not very hard to develop the template for answering such questions, but students typically miss certain requirements of specificity in their responses. Briefly stated, the first of these can be interpreted as the amount of change (in whatever units the change is measured) over the interval [*a, b*]. The second is a bit more complicated; it measures the average rate of change of the quantity, in amount per unit of time, over whatever the interval. Failure to be specific about units and intervals can be costly.

This type of question is also an opportunity to showcase the various types of representations of data – by graph, by table, or analytically – so students should be looking for any one of these to be presented on the exam.

Here once again is an example to support the idea.

Example 5:

The function $F(t)=3e^{-.2t}$ represents the rate at which water is flowing out of an outlet pipe, measured in thousands of gallons per minute, over the interval $0 \le t \le 10$.

A. How much water flowed out of the pipe during the entire 10 minute interval?
B. Using correct units, explain the meaning of $\int_0^5 F(t)\,dt$.
C. Using correct units, explain the meaning of $\frac{1}{5}\int_0^5 F(t)\,dt$.

4. Dealing with Unequal Subintervals.

For a long time on the AP Calculus exam, it was understood that values presented in a table would have been given over equal subintervals. Not only is that not the rule; in recent years, equal subintervals would be the exception.

As a rule, table values are presented when a Riemann sum or a Trapezoidal sum is required to estimate the value of a definite integral. Many students become accustomed to the formulas that are presented for these sums when there are no decisions about the size of intervals. However, remembering the formula by rote can lead students to forget that the individual terms of the sum are rectangles or trapezoids with their own dimensions. Students should have these formulas well in hand:

Let $f'(x)$ be continuous on an interval $[x_0, x_n]$ divided into n subintervals by the values $x_0, x_1, x_2, \ldots, x_n$. The left and right Riemann sum estimates for $\int_{x_0}^{x_n} f'(x)dx$ are:

Left Riemann Sum:

$$\int_{x_0}^{x_n} f'(x)dx \approx f'(x_0)\cdot(x_1-x_0)+f'(x_1)\cdot(x_2-x_1)+f'(x_2)\cdot(x_3-x_2)+\ldots+f'(x_{n-1})\cdot(x_n-x_{n-1})$$

Right Riemann Sum:

$$\int_{x_0}^{x_n} f'(x)dx \approx f'(x_1)\cdot(x_1-x_0)+f'(x_2)\cdot(x_2-x_1)+f'(x_3)\cdot(x_3-x_2)+\ldots+f'(x_n)\cdot(x_n-x_{n-1})$$

Trapezoidal Sum with Unequal Subintervals:

Let $f'(x)$ be continuous on an interval $[x_0, x_n]$ divided into n subintervals by the values $x_0, x_1, x_2, \ldots, x_n$. The trapezoidal sum estimate for $\int_{x_0}^{x_n} f'(x)dx$ is:

$$\int_{x_0}^{x_n} f'(x)dx \approx \frac{f'(x_0)+f'(x_1)}{2}\cdot(x_1-x_0)+\frac{f'(x_1)+f'(x_2)}{2}\cdot(x_2-x_1)+$$
$$\frac{f'(x_2)+f'(x_3)}{2}\cdot(x_3-x_2)+\ldots+\frac{f'(x_{n-1})+f'(x_n)}{2}\cdot(x_n-x_{n-1})$$

Note how the familiar pattern of coefficients 1,2,2,…,1 is no longer evident, since each average is taken over a possibly different sized interval.

Example 6:

Carmen entered a bicycle race, and her non-decreasing velocity, in meters per second, was registered at various times during the interval $0 \le t \le 12$, as shown in the table below. S is her speed in meters per second after t seconds. The distance Carmen bicycled is a differentiable function of time *t*.

t (Seconds)	0	3	5	9	12
S(Meters/sec)	4	6	7	10	12

A. Use the data in the table to estimate Carmen's acceleration at time *t* = 7 seconds.
B. Use a trapezoidal estimate with four subintervals to estimate how far Carmen travelled during the twelve second interval.
C. Use a left Riemann sum with four subintervals to estimate $\frac{1}{12}\int_0^{12} S(t)\,dt$. Using correct units, describe the meaning of this quantity in the context of the problem.

5. Functions Defined by Integrals.

Although there are not a great many direct applications for functions defined by integrals, questions involving this stem hold a lot of potential for asking a variety of concept questions. Thus, students may expect to see them for some time to come. There are a few key considerations when working with a function that has been defined as the integral of a given function, usually one whose graph is provided.

a) If given $F(x)=\int_a^x f(t)\,dt$, it is helpful to think of the pictured *f(x)* in the same way we consider the derivative of a function as an indicator of the behavior of the function.

b) There is almost always a question that asks for one or more of: $F(k); f(k); f'(k)$. This is a prompt for three separate realizations:
   a) Computing a function value by computing area and direction;
   b) Computing a function value by reading it from a graph; and
   c) Computing the slope of a graph at a given point.

   Giving explicit answers to these labeled and in the order requested is helpful, such as:

   $F(k)=p; f(k)=r; f'(k)=s$

   As a rule, no specific reasoning needs to be provided.

c) This is a question in which it is good to remember that relative extrema of the graph of a derivative of a function occur when the function itself has a point of inflection. This requires relating *f'(x)* to *f(x)* in the same way that *f(x)* is related to *F(x).* Often a reason is required, and students must remember to cite that the second derivative changes sign. A frequent error is to declare an extremum or point of inflection only by stating the appropriate derivative is zero, but neglecting that the derivative must change sign at that point. It is not necessary to say what sign it has on either side.

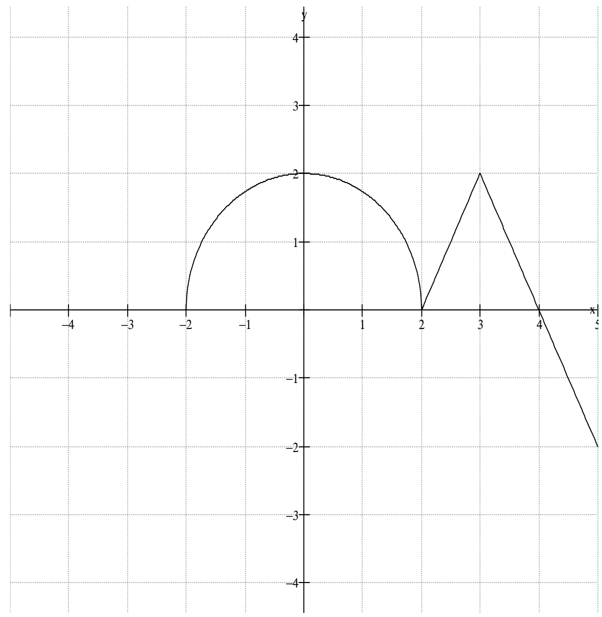
Example 7:

The continuous function *f* is defined on the interval $-2 \le x \le 5$. The graph of *f* consists of two line segments and a semicircle of radius 2. Let $F(x) = \int_{-2}^{x} f(t)\, dt$.

A. Find the values of :
   $F(0);\ f(3);\ f'(4)$

B. For which value(s) of *x* does *F* have a point of inflection?

C. At which value(s) of *x* does F have a relative maximum or minimum? Indicate which for each value and give a reason for your answer.



Graph of *f*

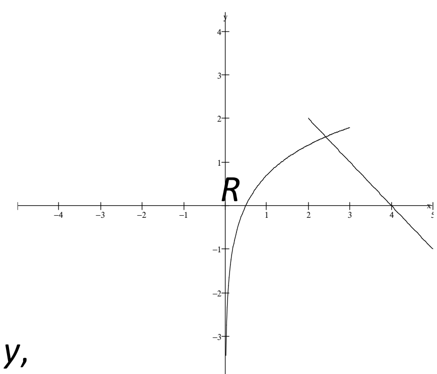
6. Volumes of Solids having Known Parallel Cross-sections.

The area and volume problem has been a staple of AP Calculus exams for decades, but recently, students have been asked to escape the formulaic to answer questions about solids that are difficult to visualize and maybe even harder to model and solve. In particular, solids that have known parallel cross sections have confronted students in recent years. A look at the recent past reveals that a section of a problem dealing with solids of this type has appeared in 2003, 2004, 2007, 2009, 2010 and 2012. In other years, variations on the typical have prevailed, including setups that are easier to do in *y*, despite the student's reflex to set them up in *x*; and also axes of revolution that are rarely the *x* or *y* axis.

There is another subtle change in play that has to do both with the mechanics of the exam and the expectations for student work. As the exam has gone to a 2-4 format as far as calculator and non-calculator problems, it is no longer automatic that the area/volume problem will appear in the early going. Since the student only has 30 minutes for these first two problems, front loading a more traditional problem – especially one in which the calculations are often carried out by calculator – is less attractive. In other words, the student who completes the first two (part A) problems in 20 minutes will burn 10 minutes waiting for part B to begin. Then, when the second part begins, the available time will be shortened in comparison to the difficulty of the problem. As a result, the expected "difficulty is proportional to cardinality" rule is not applied in free response questions.

<u>Example 8:</u>

Let *R* be a region in the plane bounded by the graphs of $f(x)=\ln(2x)$, $g(x)=4-x$ and the *x*-axis as shown.



A. Find the area of *R*.

B. The region *R* is the base of a solid. For each value of *y*, the cross section of the solid taken perpendicular to the *y*-axis is a square the length of whose side is the horizontal distance between $f(x)$ and $g(x)$. Write an integral expression for the solid and find its value.

C. A horizontal line $y=k$ divides the region *R* into two regions having equal area. Set up, but do not solve, an integral expression that can be used to find the value of *k*.

7. **Differential Equations Three Ways.**

The topic of differential equations is an attractive one because students are able to work with them using multiple representations. We can:

a) Separate variables to solve them analytically.
b) Use slope fields to show their direction graphically.
c) Use Euler's Method to approximate them numerically (BC only).

Few other topics offer such a variety of approaches. Here is more:

a) Students who enter the AP Calculus exam without a thorough grounding in how to solve differential equations using separation of variables stand a very good chance of forfeiting 5 or 6 points on the free response examination. Only twice since 2003 has there not been a differential equation solution question on free response, and in those two years (2007 and 2009) there was likely more than one multiple choice question touching on the topic. When solving a differential equation, it is good to remember these helpful tips:

   i. A good separation is essential, and any attempt to solve without separating will earn no points.
   ii. After separation, the antiderivative point(s) come quickly, as does the constant of integration. Very often, a late constant of integration - the "constant of desperation" – leads to the loss of the last three points in the problem. A timely constant, applied at the point of antidifferentiating, is most important.
   iii. Working with exponentiation can be a bit tricky at times, especially when it is compounded by an absolute value. The distinction between e^{kt+c} and Ce^{kt} is pivotal here, inasmuch as the first is always positive and the second need not be.

b) Slope fields have made an appearance on both parts of recent exams, and, although the point value for creating a slope field is often two points, they are an easy two points that cannot be ignored. A few hints that will make sure slope fields earn their credit:

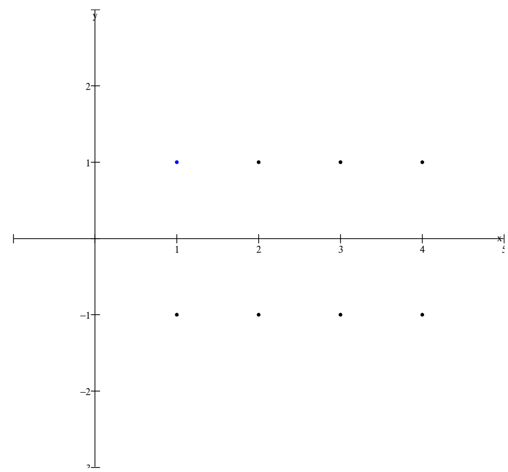
   i. Slope field segments are straight line segments, and should not be drawn as curves.
   ii. Slope field segments should be long enough to observe and short enough so that they do not interfere with other slope field segments.
   iii. If segments along a vertical or horizontal line should have decreasing or increasing slope, the relative sizes should reflect that fact.

iv. If the differential equation is provided for the slope field segments, look for the zeroes of the differential equation to give important clues. For instance, if we start with $\frac{dy}{dx} = y(x-1)$, we know that the slope segments are horizontal when $y = 0$ and when $x = 1$.

Example 9:

Consider the differential equation $\frac{dy}{dx} = y(1-x)$.

A. On the axes at right, sketch the slope field for the eight points indicated.



B. Find the particular solution $y = f(x)$ to The differential equation given that $f(2) = -1$.

C. Use Euler's Method with two steps of equal size to approximate $f(3)$, starting with $f(2) = -1$.

8. Analyzing Functions Using the Derivative Tests.

Following the emphasis that is placed on multiple representations, problems involving the analysis of functions are staples on the free response exam. While at one time these problems were almost always presented in analytical form – given a function $f(x)$, use the derivative(s) to determine where the function is increasing/decreasing, concave up/down, and so on – it is far more common to see the problem presented now by showing the graph or a table of values for the first derivative; or, as occurred in 2005, providing a table that reveals the information about a function and its derivatives and asks for the conclusions from that information. The focus is squarely on knowing what the first and second derivative tests are and how they help us to draw conclusions about the behavior of a function.

There are several issues around this topic that students need to be aware of and able to respond to effectively. Following is a discussion of these issues.

(1) The first is the issue of sign charts. For years, students have used sign charts to help them analyze derivatives and thus to determine whether a function was increasing, decreasing or neither, and to determine the location of local extrema and points of inflection. For quite some time, the sign chart itself was accepted as justification for these phenomena, but no longer. The sign chart is perfectly okay as a starting point, but now students must supplement the information contained in the chart with calculus statements that appeal to the derivative. If a sign chart reveals that the derivative $f'(x)$ is positive on the left of $x=a$ and negative on the right of $x=a$, it is necessary to support the conclusion that *f* has a relative maximum there by saying either "because *f′* changes from positive to negative at $x=a$" or, equivalently, "$f'(x)>0$ for $x<a$ and $f'(x)<0$ for $x>a$." The student must make an explicit appeal to the derivative test as the source of this knowledge.

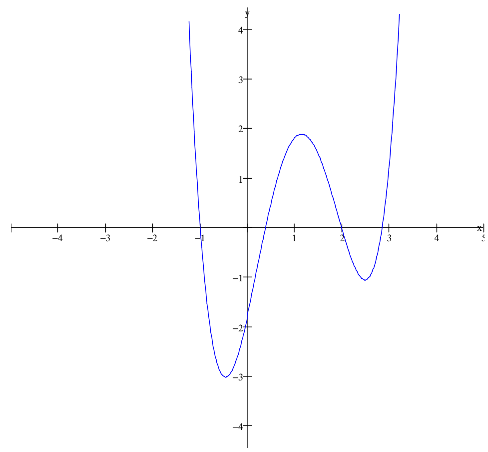
(2) Of less importance but still a subject of questions is whether intervals of increase or decrease should be open or closed. The answer is that either response has been accepted, since the interpretation relies to some extent on how these ideas of increase and decrease are treated in various textbooks. For most texts, the property of increasing-ness or decreasing-ness is one that pertains to intervals. Particles do not increase or decrease at a point, but over an interval of time. As such, if we define a function increasing on an interval in this way:

f is increasing on an interval [m, n] if, for any two values a and b in the interval,
$f(a) < f(b) \rightarrow a < b$

then we are obliged to accept that the possible values of *a* and *b* include *m* and *n*. For the present, one need not worry about this subtle distinction, but it makes for an interesting discussion of the meaning of increase and decrease in the context of points and intervals.

(3) A third issue is the often overlooked fact that extrema of the first derivative are indicators of points of inflection on the original function. A quick investigation of the derivative tells us that, when the derivative is increasing, its derivative (that is, the second derivative) is positive; and when the derivative is decreasing, its derivative is negative. This is how we detect points of inflection; but just observing extrema on the first derivative makes that observation simpler.

(4) Finally, there is the problem of not sufficiently supporting extrema and points of inflection with the appropriate discussion of neighborhoods. Students often believe that it is enough to say that extrema and points of inflection occur at the zeroes of the first and second derivative, neglecting to add that there must be an attendant sign change to close the deal.



Example 10: The graph of $f'(x)$, the derivative of $f(x)$, Is pictured at right for -2 < *x* < 4 . The graph of $f'(x)$ has zeroes where . $x = -1, .3, 2$ and 2.8 Also, $f'(x)$ has a horizontal tangent line where $x = -.3, 1.2$ and 2.5.

A. At what value(s) of *x* does $f(x)$ have a relative maximum? Justify your answer.

B. At what value(s) of *x* does $f(x)$ have a relative minimum? Justify your answer.

C. At what value(s) of *x* does $f(x)$ have a point of inflection? Justify your answer.

Solutions and discussion.

Solution 1: $\lim_{x\to 2^-}\left(\cos(\frac{\pi}{4}x)+1\right)=\cos\left(\frac{\pi}{2}\right)+1=1$

$\lim_{x\to 2^+}\left(\frac{x^2}{2}+x-3\right)=2+2-3=1$

Also, $f(2)=1$

So, $\lim_{x\to 2} f(x)=f(2)$

And *f* is continuous at $x=2$

Solution 2:

$v(4)=.172>0$
$a(4)=v'(4)=-.293<0$

The speed is decreasing at time $t=4$ since v and a have opposite signs.

Solution 3:

A. The distance travelled is simply the value of the definite integral:

$\int_0^2 |(t-2)\sin(e^t)|\, dt$ = 1.582

B. The average velocity is the distance divided by the interval length:

$\frac{1}{2-0}\int_0^2 |(t-2)\sin(e^t)|\, dt$ = .791

Solution 4: We begin with the relationship: $x(b)=k+\int_a^b v(t)\,dt$ and enter the information we have been given, so that: $x(3)=x(0)+\int_0^3 \sin(t^2)\,dt=4+.774=4.774$

Solution 5:

A.

$$\int_0^{10} F(t)\,dt = 12.970 \text{ thousand gallons}$$

B.

$\int_0^{5} F(t)\,dt$ represents the total amount of water that flowed out of the pipe from $t = 0$ to $t = 5$ minutes, measured in thousands of gallons.

C.

$\frac{1}{5}\int_0^{5} F(t)\,dt$ represents the average rate, in thousands of gallons per minute, at which water flowed out of the outlet pipe during the first 5 minutes of the interval.

[Notice that in the third part the interval was described as the first 5 minutes. Since the interval was stated to be [0, 10], this is fine. We could also have said "from $t = 0$ to $t = 5$." Also, note that in the last two parts, there was no requirement to calculate the values, even though that could have been done. A value without a proper explanation would not have earned credit.]

Solution 6:

A. Acceleration $\approx \dfrac{S(9)-S(5)}{9-5} = \dfrac{3}{4}\, m/s^2$

B.

$$\text{Distance } \approx 3\left(\frac{S(0)+S(3)}{2}\right)+2\left(\frac{S(3)+S(5)}{2}\right)+4\left(\frac{S(5)+S(9)}{2}\right)+3\left(\frac{S(9)+S(12)}{2}\right)$$

$$= 95 \text{ meters}$$

C. $\dfrac{1}{12}\displaystyle\int_0^{12} S(t)\,dt \approx 6.833\, m/s$. This is an estimate of Carmen's average velocity, in meters per second, over the entire 12 second interval.

Solution 7:

A. $F(0)=\pi$; $f(3)=2$; $f'(4)=-2$

B. F has a point of inflection where $x=0, 2$ and 3 since $F''(x)=f'(x)$ changes sign there.

C. F has a relative maximum at $x=4$ only, since $f=F'$ changes from positive to negative there. There are no other relative extrema of F.

Solution 8:

f and g intersect at (a,b)=(2.422183, 1.577817)

A. Solving *f* and *g* respectively for *x* in terms of *y*, we have:

$x=.5e^y$ *and* $x=4-y$; therefore,

$$A=\int_0^b\left((4-y)-.5e^y\right)dy=3.144$$

B.

$$V=\int_0^b\left[(4-y)-(.5e^y)\right]^2 dy=7.827$$

[Notice here that the student is forced to use a function in *y*.]

C.

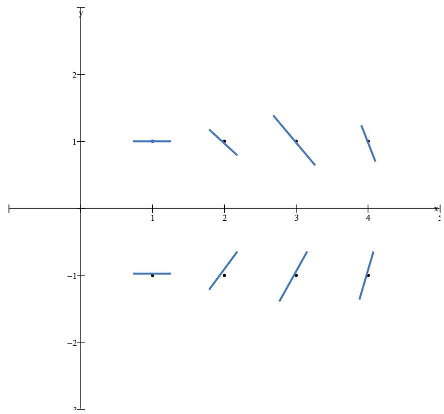
$$\int_0^k\left[(4-y)-(.5e^y)\right]dy=\int_k^b\left[(4-y)-(.5e^y)\right]dy$$

or

$$\int_0^k\left[(4-y)-(.5e^y)\right]dy=\frac{1}{2}(7.827) \text{ or } 3.913$$

Solution 9:

A.



B.

$$\frac{dy}{dx} = y(1-x)$$

$$\frac{1}{y}dy = (1-x)dx$$

$$\ln|y| = x - \frac{1}{2}x^2 + C_1$$

$$|y| = e^{x-\frac{1}{2}x^2+C_1}$$

$$y = Ce^{x-\frac{1}{2}x^2}$$

$$-1 = Ce^0 \rightarrow C = -1$$

$$y = -e^{x-\frac{1}{2}x^2}$$

C.

$$f(2.5) \approx f(2) + f'(2)(.5)$$

$$= -1 + 1(.5)$$

$$= -.5$$

$$f(3) \approx f(2.5) + f'(2.5)(.5)$$

$$= -.5 + (.75)(.5)$$

$$= -.125$$

Solution 10:

A. *f* has a relative maximum at *x* = 1 and *x* = 2 because $f'(x)$ changes sign from positive to negative.

B. *f* has a relative minimum at *x* = .3 and *x* = 2.8 because $f'(x)$ changes sing from negative to positive.

C. *f* has a point of inflection at $x = -1, .3, 2 \text{ and } 2.8$ because *f''*(*x*) changes sign at those values.